

Chapter 1: Limits and Their Properties

Section 1.1: A Preview of Calculus

Calculus (often referred to as *the calculus*) is a branch of mathematics that deals with continuously varying quantities. Its invention is traditionally ascribed to two 17th century mathematicians: Gottfried Wilhelm Leibniz and Isaac Newton. However, many of the key concepts of the calculus had been explored well before their times. For instance, the idea of the integral had been studied by Archimedes in the third century BCE. Newton and Leibniz have the distinction, however, of being the first to synthesize the two major halves of calculus: the differential calculus and the integral calculus.

Differential calculus is related to the problem of finding the slope of an curve at a particular point, while the idea of integral calculus came from efforts to calculate the area under curves. Newton and Leibniz showed that these apparently unrelated ideas are in fact intimately related by a statement known as the *fundamental theorem of calculus*. (This very important concept is introduced in section 4.4.) Pages 43 and 44 show several concepts which can be analyzed using calculus, but for which algebraic techniques are insufficient.

Section 1.2: Finding Limits Graphically and Numerically

Limits

If a function $f(x)$ approaches a particular value L when x gets arbitrarily close to a particular number c , the function is said to have a *limit* at c equal to L . Mathematically, this statement is written

$$\lim_{x \rightarrow c} f(x) = L$$

For example, consider the function $f(x) = 2x$, calculating limits is straightforward. One can see that as x gets close arbitrarily to 3, the function $f(x) = 2x$ gets arbitrarily close to 6. In other words, $\lim_{x \rightarrow 3} 2x = 6$.

It is not necessary for the function to exist at the point where the limit is evaluated. For example, the function $f(x) = \frac{x^3 - 1}{x - 1}$ does not exist at the point $x = 1$, because $f(1)$ results in a division by zero. However, the limit of $f(x)$ at as x approaches 1 does exist and is equal to 3. We can see this by plugging in values of x that are successively closer to 1:

x	0.75	0.9	0.99	0.999
$f(x)$	2.313	2.710	2.970	2.997

x	1.25	1.1	1.01	1.001
$f(x)$	3.813	3.310	3.030	3.003

Nonexistent limits

The function $f(x) = \frac{|x|}{x}$ does *not* have a limit at $x = 0$, because the function does not approach a particular value. Instead, as x approaches 0 from the right, $f(x)$ approaches 1, while as x approaches 0 from the left, $f(x)$ approaches -1 (see **Figure 1**).

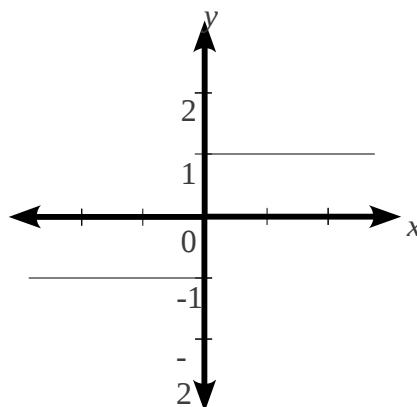


Figure 1: The function $f(x) = \frac{|x|}{x}$ does not have a limit as x approaches 0.

Another limit that does not exist is $\lim_{x \rightarrow 0} \frac{1}{x^2}$. In this case, the function does not approach any particular value, but instead becomes arbitrarily large near small values of x .

Formal definition of limits

To be more precise, suppose that c varies by a small amount δ and the corresponding L value varies by a corresponding amount ϵ as shown in **Figure 2**. If the limit exists, you must be able to make ϵ as *small as you want* simply by decreasing the size of δ .

$\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist, because no matter how close x gets to 0, the difference between the corresponding y -values is always equal to 1.

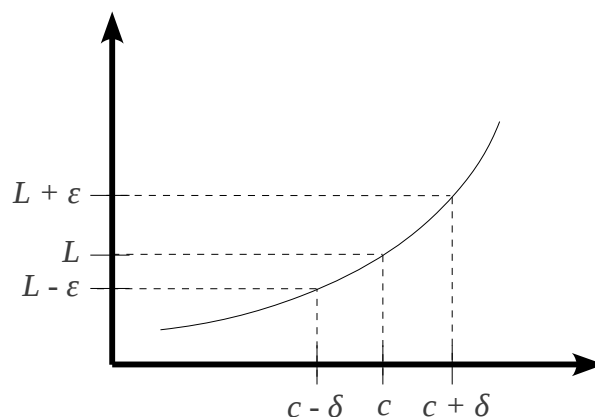


Figure 2: Formal definition of limits

Section 1.3: Evaluating Limits Analytically

For “normal” functions like $f(x) = 2x$, calculating limits is straightforward. All one needs to do is plug the appropriate value of x into the function. For example, one can show that

$$\lim_{x \rightarrow 3} 2x = 6, \text{ just by plugging } x = 3 \text{ into the function } f(x) = 2x.$$

However, there are other limits for which one cannot simply plug the x value into the function, because the function does not exist at that particular x -value, or because the value of the function at that point and the value it is approaching are different. Not surprisingly, it is these special cases that we are most interested in.

Often there is some algebraic “trick” one can use to evaluate limits.

For example, the limit $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$ may be evaluated by factoring the numerator and

canceling out the denominator. Once the problematic $x - 1$ term has been canceled out, it is possible to plug in $x = 1$ into the remainder.

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \rightarrow 1} (x^2 + x + 1) = 3$$

Another useful trick is “multiplying by the conjugate.” For example,

$$\lim_{x \rightarrow 3} \frac{\sqrt{x + 1} - 2}{x - 3} = \lim_{x \rightarrow 3} \frac{\sqrt{x + 1} - 2}{x - 3} \cdot \frac{\sqrt{x + 1} + 2}{\sqrt{x + 1} + 2} = \lim_{x \rightarrow 3} \frac{x - 3}{(x - 3)(\sqrt{x + 1} + 2)} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x + 1} + 2} = \frac{1}{4}$$

Squeeze theorem

The *squeeze theorem* states that if one function always lies between two functions over an open interval containing c , and that the limit both of the outside functions at c is L , then the limit of the middle function must also be L (see **Figure 3**).

The squeeze theorem can be used to prove two trigonometric limits of particular importance:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

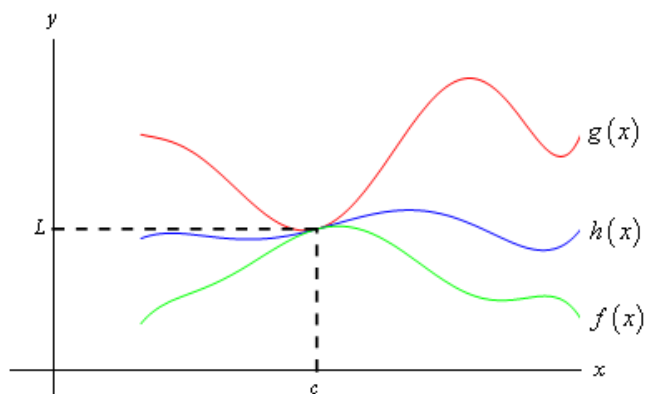


Figure 3: $h(x)$ lies between $f(x)$ and $g(x)$. Because the limits of $f(x)$ and $g(x)$ as x approaches c are L , the limit of $h(x)$ as x approaches c must also be L .